

Non-minimal Wu-Yang wormhole

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We discuss exact solutions of three-parameter non-minimal Einstein-Yang-Mills model, which describe the wormholes of a new type. These wormholes are considered to be supported by SU(2)-symmetric Yang-Mills field, non-minimally coupled to gravity, the Wu-Yang ansatz for the gauge field being used. We distinguish between regular solutions, describing traversable non-minimal Wu-Yang wormholes, and black wormholes possessing one or two event horizons. The relation between the asymptotic mass of the regular traversable Wu-Yang wormhole and its throat radius is analysed.

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I. INTRODUCTION

Wormholes are topological handles in spacetime linking widely separated regions of a single universe, or “bridges” joining two different spacetimes [1]. Recent interest in these configurations has been initiated by Morris and Thorne [2]. These authors constructed and investigated a class of objects they referred to as “traversable wormholes”.

The central feature of wormhole physics is the fact that traversable wormholes are accompanied by an unavoidable violation of the null energy condition, i.e., the matter threading the wormhole’s throat has to be possessed of “exotic” properties [2, 3]. The known classical matter does satisfy the usual energy conditions, hence physical models providing the existence of wormholes must include hypothetical forms of matter. Various models of such kind have been considered in the literature, among them scalar fields [4]; wormhole solutions in semi-classical gravity [5]; solutions in Brans-Dicke theory [6]; wormholes on the brane [7]; wormholes supported by matter with an exotic equation of state, namely, phantom energy [8], the generalized Chaplygin gas [9], tachyon matter [10], etc [11, 12].

The electromagnetic field and the non-Abelian gauge field can also be considered as sources for wormholes when they satisfy the necessary unusual energy conditions. Such possibility can in principle appear if one considers nonlinear electrodynamics [13] or takes into account the non-minimal coupling of gravity with vector-type fields, i.e., with the non-Abelian Yang-Mills field, or Maxwell field. The non-minimal Einstein-Maxwell theory has been elaborated in detail in both linear (see, e.g., [14, 15] for a review) and non-linear (see, [16]) versions. As for the non-minimal Einstein-Yang-Mills theory, two concepts to derive the master equations are known. The first one is a dimensional reduction of the Gauss-Bonnet action [17], this model contains one coupling parameter. The second concept is a non-Abelian generalization of the non-minimal non-linear Einstein-Maxwell theory [16]. We will follow the latter approach and construct a three-parameter non-minimal model being linear in the curvature by analogy with the well-known model proposed by Drummond and Hathrell for linear electrodynamics [18]. Three coupling constants q_1 , q_2 , and q_3 of the model are shown to introduce a new specific radius associated with the radius a of the wormhole throat.

In this work we focus on the example of exact solution of the non-minimal three-parameter EYM model describing the wormhole of a new type, namely, *non-minimal wormhole*. It can also be indicated as *non-minimal Wu-Yang wormhole*, since the solution of the non-minimally extended Yang-Mills subsystem of the total self-consistent EYM system of equations is the direct analog of the Wu-Yang monopole [19].

The paper is organized as follows. In Sec. II we briefly describe the formalism of three-parameter non-minimal Einstein-Yang-Mills model. In Sec. III, Subsect. A we adapt this model for the case of static spherically symmetric field configuration, present the exact solution of the Wu-Yang type to the gauge field equations and formulate two key equations for two metric functions $\sigma(r)$ and $N(r)$. In Subsect. B, C and D we discuss the details of the three-parameter family of exact solutions for the function $\sigma(r)$. Sec. IV is devoted to the analysis of the solution describing the non-minimal Wu-Yang wormhole. Conclusions are formulated in the last section.

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II. NON-MINIMAL EINSTEIN-YANG-MILLS MODEL

The action of the three-parameter non-minimal Einstein-Yang-Mills model has the form¹

$$S_{\text{NMEYM}} = \int d^4x \sqrt{-g} \left(\frac{1}{8\pi} R + \frac{1}{2} F_{ik}^{(a)} F^{ik(a)} + \frac{1}{2} \chi^{ikmn} F_{ik}^{(a)} F_{mn}^{(a)} \right), \quad (1)$$

where $g = \det(g_{ik})$ is the determinant of a metric tensor g_{ik} , and R is the Ricci scalar. The Latin indices without parentheses run from 0 to 3, the summation with respect to the repeated group indices (a) is implied. The tensor χ^{ikmn} , indicated in [16] as non-minimal susceptibility tensor, is defined as follows:

$$\chi^{ikmn} \equiv \frac{q_1}{2} R (g^{im} g^{kn} - g^{in} g^{km}) + \frac{q_2}{2} (R^{im} g^{kn} - R^{in} g^{km} + R^{kn} g^{im} - R^{km} g^{in}) + q_3 R^{ikmn}. \quad (2)$$

Here R^{ik} and R^{ikmn} are the Ricci and Riemann tensors, respectively, and q_1, q_2, q_3 are the phenomenological parameters describing the non-minimal coupling of the Yang-Mills and gravitational fields. Following [20], we consider the Yang-Mills field, \mathbf{F}_{mn} , to take the values in the Lie algebra of the gauge group $SU(2)$:

$$\mathbf{F}_{mn} = -i\mathcal{G}F_{mn}^{(a)} \mathbf{t}_{(a)}, \quad \mathbf{A}_m = -i\mathcal{G}A_m^{(a)} \mathbf{t}_{(a)}. \quad (3)$$

Here $\mathbf{t}_{(a)}$ are Hermitian traceless generators of $SU(2)$ group, \mathcal{G} is a constant of gauge interaction, and the group index (a) runs from 1 to 3. The generators $\mathbf{t}_{(a)}$ satisfy the commutation relations:

$$[\mathbf{t}_{(a)}, \mathbf{t}_{(b)}] = i\varepsilon_{(a)(b)(c)} \mathbf{t}_{(c)}, \quad (4)$$

where $\varepsilon_{(a)(b)(c)}$ is the completely antisymmetric symbol with $\varepsilon_{(1)(2)(3)} = 1$. The Yang-Mills field potential, \mathbf{A}_i , and strength field, \mathbf{F}_{ik} , are coupled by the relation

$$\mathbf{F}_{ik} = \partial_i \mathbf{A}_k - \partial_k \mathbf{A}_i + [\mathbf{A}_i, \mathbf{A}_k], \quad (5)$$

which guarantees that the equation

$$\hat{D}_l \mathbf{F}_{ik} + \hat{D}_k \mathbf{F}_{li} + \hat{D}_i \mathbf{F}_{kl} = 0 \quad (6)$$

turns into identity. Here the symbol \hat{D}_k denotes the gauge invariant derivative

$$\hat{D}_i \equiv \nabla_i + [\mathbf{A}_i, \quad], \quad (7)$$

and ∇_m is a covariant spacetime derivative.

The variation of the action (1) with respect to Yang-Mills potential $A_i^{(a)}$ yields

$$\hat{D}_k \mathbf{H}^{ik} \equiv \nabla_k \mathbf{H}^{ik} + [\mathbf{A}_k, \mathbf{H}^{ik}] = 0, \quad \mathbf{H}^{ik} = \mathbf{F}^{ik} + \chi^{ikmn} \mathbf{F}_{mn}. \quad (8)$$

The tensor \mathbf{H}^{ik} is a non-Abelian analog of the induction tensor known in the electrodynamics [21], and thus χ^{ikmn} can be considered as a non-minimal susceptibility tensor [16]. The variation of the action with respect to the metric g_{ik} yields

$$R_{ik} - \frac{1}{2} R g_{ik} = 8\pi T_{ik}^{(\text{eff})}. \quad (9)$$

The effective stress-energy tensor $T_{ik}^{(\text{eff})}$ can be divided into four parts:

$$T_{ik}^{(\text{eff})} = T_{ik}^{(YM)} + q_1 T_{ik}^{(I)} + q_2 T_{ik}^{(II)} + q_3 T_{ik}^{(III)}. \quad (10)$$

The first term $T_{ik}^{(YM)}$:

$$T_{ik}^{(YM)} \equiv \frac{1}{4} g_{ik} F_{mn}^{(a)} F^{mn(a)} - F_{in}^{(a)} F_k^{n(a)}, \quad (11)$$

¹ Hereafter we use the units $c = G = \hbar = 1$.

is a stress-energy tensor of the pure Yang-Mills field. The definitions of other three tensors relate to the corresponding coupling constants q_1, q_2, q_3 :

$$T_{ik}^{(I)} = R T_{ik}^{(YM)} - \frac{1}{2} R_{ik} F_{mn}^{(a)} F^{mn(a)} + \frac{1}{2} [\hat{D}_i \hat{D}_k - g_{ik} \hat{D}^l \hat{D}_l] [F_{mn}^{(a)} F^{mn(a)}], \quad (12)$$

$$\begin{aligned} T_{ik}^{(II)} = & -\frac{1}{2} g_{ik} \left[\hat{D}_m \hat{D}_l \left(F^{mn(a)} F_n^{l(a)} \right) - R_{lm} F^{mn(a)} F_n^{l(a)} \right] \\ & - F^{ln(a)} \left(R_{il} F_{kn}^{(a)} + R_{kl} F_{in}^{(a)} \right) - R^{mn} F_{im}^{(a)} F_{kn}^{(a)} - \frac{1}{2} \hat{D}^m \hat{D}_m \left(F_{in}^{(a)} F_k^{n(a)} \right) \\ & + \frac{1}{2} \hat{D}_l \left[\hat{D}_i \left(F_{kn}^{(a)} F^{ln(a)} \right) + \hat{D}_k \left(F_{in}^{(a)} F^{ln(a)} \right) \right], \end{aligned} \quad (13)$$

$$\begin{aligned} T_{ik}^{(III)} = & \frac{1}{4} g_{ik} R^{mnls} F_{mn}^{(a)} F_{ls}^{(a)} - \frac{3}{4} F^{ls(a)} \left(F_i^{n(a)} R_{knls} + F_k^{n(a)} R_{inls} \right) \\ & - \frac{1}{2} \hat{D}_m \hat{D}_n \left[F_i^{n(a)} F_k^{m(a)} + F_k^{n(a)} F_i^{m(a)} \right]. \end{aligned} \quad (14)$$

The tensor $T_{ik}^{(\text{eff})}$ satisfies the conservation law $\nabla^k T_{ik}^{(\text{eff})} = 0$. The self-consistent system of equations (8) and (9) with (10)-(14) is a direct non-Abelian generalization of the three-parameter non-minimal Einstein-Maxwell model discussed in [16]. This system can also be considered as one of the variants of a non-minimal generalization of the Einstein-Yang-Mills model.

III. EXACT SOLUTIONS OF THE STATIC MODEL WITH SPHERICAL SYMMETRY

A. Master equations

Let us take the metric of a static spherically symmetric spacetime in the form to be especially convenient for studying a wormhole geometry:

$$ds^2 = \sigma^2 N dt^2 - \frac{dr^2}{N} - (r^2 + a^2) (d\theta^2 + \sin^2 \theta d\varphi^2), \quad (15)$$

where the metric functions σ and N depend only on r . The properties of *traversable* wormholes dictate some additional requirements for the metric (15), which were in great detail discussed in [1, 2]. In particular, we note that

- (i) the radial coordinate r runs from $-\infty$ to $+\infty$. Two asymptotical regions $r = -\infty$ and $r = +\infty$ are connected by the wormhole's throat which has the radius a and is located at $r = 0$.
- (ii) Since the spacetime of a traversable wormhole has neither singularities nor event horizons, the metric components $g_{tt} = \sigma^2 N$ and $-g_{rr} = 1/N$ should be regular and positive everywhere. Note that, in particular, this means that $N(r)$ is positive defined, both $\sigma(r)$ and $N(r)$ are finite, and neither $\sigma(r)$ nor $N(r)$ can take zero values.
- (iii) In addition, one may demand the asymptotical flatness of the wormhole spacetime at $r = \pm\infty$. This is guaranteed provided the following boundary conditions for the functions σ and N are satisfied:

$$\sigma^2(\pm\infty) = 1, \quad N(\pm\infty) = 1. \quad (16)$$

Below we will search for solutions of the non-minimal Einstein-Yang-Mills model, which satisfy the listed requirements.

The non-minimal Yang-Mills equations (8) are satisfied identically, when the gauge field is parameterized as [22, 23]

$$\mathbf{A}_0 = \mathbf{A}_r = 0, \quad \mathbf{A}_\theta = i\mathbf{t}_\varphi, \quad \mathbf{A}_\varphi = -i\nu \sin \theta \mathbf{t}_\theta, \quad (17)$$

which is known to be the so-called Wu-Yang monopole solution [19]. The parameter ν is a non-vanishing integer, \mathbf{t}_r , \mathbf{t}_θ and \mathbf{t}_φ are the position-dependent generators of the SU(2) group:

$$\mathbf{t}_r = \cos \nu \varphi \sin \theta \mathbf{t}_{(1)} + \sin \nu \varphi \sin \theta \mathbf{t}_{(2)} + \cos \theta \mathbf{t}_{(3)},$$

$$\mathbf{t}_\theta = \partial_\theta \mathbf{t}_r, \quad \mathbf{t}_\varphi = \frac{1}{\nu \sin \theta} \partial_\varphi \mathbf{t}_r, \quad (18)$$

which satisfy the following commutation rules

$$[\mathbf{t}_r, \mathbf{t}_\theta] = i \mathbf{t}_\varphi, \quad [\mathbf{t}_\theta, \mathbf{t}_\varphi] = i \mathbf{t}_r, \quad [\mathbf{t}_\varphi, \mathbf{t}_r] = i \mathbf{t}_\theta. \quad (19)$$

The field strength tensor \mathbf{F}_{ik} has only one non-vanishing component

$$\mathbf{F}_{\theta\varphi} = i\nu \sin \theta \mathbf{t}_r. \quad (20)$$

Since the effective energy-momentum tensor $T_{ik}^{(\text{eff})}$ is divergence-free, the Einstein equations for the spherical symmetric metric (15) are known to be effectively reduced to the two key equations, say, for equations with $i = k = 0$ and $i = k = r$. The components G_0^0 and G_r^r of the Einstein tensor $G_i^k = R_i^k - \frac{1}{2}\delta_i^k R$ are

$$G_0^0 = \frac{1 - rN' - N}{r^2 + a^2} - \frac{Na^2}{(r^2 + a^2)^2}, \quad (21)$$

$$G_r^r = \frac{1 - rN' - N}{r^2 + a^2} - \frac{2rN\sigma'}{\sigma(r^2 + a^2)} + \frac{a^2N}{(r^2 + a^2)^2}. \quad (22)$$

The corresponding components of the effective energy-momentum tensor (see (11)-(14)) take the form

$$T_0^0 = \frac{\nu^2}{\mathcal{G}^2} \left(\frac{1}{2(r^2 + a^2)^2} - \frac{2q_1Na^2}{(r^2 + a^2)^4} - \frac{q_1N'r + q_1 + q_2 + q_3}{(r^2 + a^2)^3} + \frac{(13q_1 + 4q_2 + q_3)Nr^2}{(r^2 + a^2)^4} \right), \quad (23)$$

$$T_r^r = \frac{\nu^2}{\mathcal{G}^2} \left[\frac{1}{2(r^2 + a^2)^2} - \frac{1}{(r^2 + a^2)^3} \left(q_1N'r + q_1 + q_2 + q_3 + \frac{2q_1Nr\sigma'}{\sigma} \right) - \frac{(7q_1 + 4q_2 + q_3)Nr^2}{(r^2 + a^2)^4} \right]. \quad (24)$$

The difference $G_0^0 - G_r^r = 8\pi (T_0^0 - T_r^r)$ of these equations can be transformed into the decoupled equation for $\sigma(r)$ only

$$r \left[1 - \frac{\kappa q_1}{(r^2 + a^2)^2} \right] \frac{\sigma'}{\sigma} = \frac{a^2}{(r^2 + a^2)} + \frac{\kappa}{(r^2 + a^2)^3} [(10q_1 + 4q_2 + q_3)r^2 - q_1a^2], \quad (25)$$

where κ is a new charge parameter with the dimensionality of area, $\kappa = 8\pi\nu^2/\mathcal{G}^2$. Solving this equation one can find the function $\sigma(r)$ in the explicit form for arbitrary values of parameters q_1 , q_2 and q_3 :

$$\sigma(r) = \frac{r}{\sqrt{r^2 + a^2}} \left[1 - \frac{\kappa q_1}{(r^2 + a^2)^2} \right]^{\frac{10q_1 + 4q_2 + q_3}{4q_1}}. \quad (26)$$

Excluding the function σ from the Einstein equation with $i = k = 0$, we obtain the equation for $N(r)$ only:

$$\begin{aligned} r \left[1 - \frac{\kappa q_1}{(r^2 + a^2)^2} \right] N' + N \left[1 + \frac{a^2}{(r^2 + a^2)} + \frac{\kappa(13q_1 + 4q_2 + q_3)}{(r^2 + a^2)^2} - \frac{\kappa a^2(15q_1 + 4q_2 + q_3)}{(r^2 + a^2)^3} \right] = \\ = 1 - \frac{\kappa}{2(r^2 + a^2)} + \frac{\kappa(q_1 + q_2 + q_3)}{(r^2 + a^2)^2}. \end{aligned} \quad (27)$$

The solution of Eq. (27) can be clearly represented in quadratures. Note that the coefficient $\Theta(r) \equiv r [1 - \kappa q_1(r^2 + a^2)^{-2}]$ in front of the first (highest) derivative in both differential equations (25) and (27) can take, in principle, zero values depending on the sign of the guiding parameter q_1 . Thus, searching for the solutions of these equations, we have to distinguish three qualitatively different cases $q_1 < 0$, $q_1 = 0$ and $q_1 > 0$.

B. The case $q_1 < 0$

For negative q_1 the coefficient $\Theta(r)$ has only one root, $r = 0$. In this case, the function $\sigma(r)$ takes the form

$$\sigma(r) = \frac{r}{\sqrt{r^2 + a^2}} \left[1 + \frac{\kappa|q_1|}{(r^2 + a^2)^2} \right]^{\frac{10|q_1| - 4q_2 - q_3}{4|q_1|}}. \quad (28)$$

Notice that $\sigma(r)$ given by Eq. (28) turns into zero at $r = 0$, i.e., $\sigma(0) = 0$. This violates the condition (ii), and so this solution cannot describe a traversable wormhole. Note also that $\sqrt{-g(0)} = 0$, where $g = -\sigma^2(r^2 + a^2)^2 \sin^2 \theta$ is the determinant of g_{ik} . This means that the chosen coordinate system is ill-defined at $r = 0$, and well-defined only in the range $(0, +\infty)$ (or, equivalently, in $(-\infty, 0)$).

C. The case $q_1 = 0$

In this case $\Theta(r) = r$, and so $r = 0$ is the only root of $\Theta(r)$ as in the case $q_1 < 0$. The solution for $\sigma(r)$ transforms now into

$$\sigma(r) = \frac{r}{\sqrt{r^2 + a^2}} \exp \left\{ -\frac{\kappa(4q_2 + q_3)}{(r^2 + a^2)^2} \right\}. \quad (29)$$

The function $\sigma(r)$ given by Eq. (29) turns into zero at $r = 0$, i.e., $\sigma(0) = 0$. This means that the case $q_1 = 0$ does not admit the existence of traversable wormholes.

D. The case $q_1 > 0$

For positive q_1 the number of real roots of $\Theta(r)$ depends on the value $\beta \equiv (\kappa q_1)^{1/4}$. In case $\beta > a$, $\Theta(r)$ has three real roots, namely, $r = 0$, $r = \pm r^*$, where

$$r^* \equiv \sqrt{\beta^2 - a^2}. \quad (30)$$

For $\beta = a$ the roots $r = \pm r^*$ coincide with $r = 0$, and for $\beta < a$ one has only one real root $r = 0$. Below we consider each case separately.

I. $\beta < a$. Rewrite the solution (26) as follows

$$\sigma(r) = \frac{r}{\sqrt{r^2 + a^2}} \left[\frac{(r^2 + a^2 - \beta^2)(r^2 + a^2 + \beta^2)}{(r^2 + a^2)^2} \right]^{\frac{10q_1 + 4q_2 + q_3}{4q_1}}. \quad (31)$$

It is clear that due to the condition $\beta < a$ the expression in square brackets in Eq. (31) is positive for all r . Thus, for all values of the power parameter $(10q_1 + 4q_2 + q_3)/4q_1$ the sign of the function $\sigma(r)$ inherits the sign of r , and turns into zero at $r = 0$, i.e., $\sigma(0) = 0$. As in previous cases, this means that traversable wormholes do not exist.

II. $\beta > a$. In this case the expression in square brackets in Eq. (31) vanishes, when $r = r^* \equiv (\beta^2 - a^2)^{1/2}$. Now, depending on the sign of the power parameter $(10q_1 + 4q_2 + q_3)/4q_1$, the solution $\sigma(r)$ turns into zero or tends to infinity at r^* . When $10q_1 + 4q_2 + q_3 = 0$, one has again $\sigma(0) = 0$. Thus, the case $\beta > a$ also does not admit traversable wormholes.

III. $\beta = a$ (or, equivalently, $q_1 = a^4/\kappa$). It will be convenient to rewrite the solution (31) for $\sigma(r)$ in the following form:

$$\sigma(r) = \sqrt{\frac{r^2 + a^2}{r^2 + 2a^2}} \left[\frac{r^2(r^2 + 2a^2)}{(r^2 + a^2)^2} \right]^{\frac{12q_1 + 4q_2 + q_3}{4q_1}}. \quad (32)$$

Now the critical point of interest is $r = 0$. The behavior of $\sigma(r)$ near $r = 0$ essentially depends on the sign of the new power parameter, namely, $12q_1 + 4q_2 + q_3$. In particular, for $12q_1 + 4q_2 + q_3 > 0$ one has $\sigma(0) = 0$, and for $12q_1 + 4q_2 + q_3 < 0$ one has $\sigma(0) = \infty$. Such behavior of $\sigma(r)$ excludes traversable wormholes. Consider the last particular case, when this parameter vanishes, $12q_1 + 4q_2 + q_3 = 0$. Now we obtain

$$\sigma(r) = \sqrt{\frac{r^2 + a^2}{r^2 + 2a^2}}. \quad (33)$$

The function $\sigma(r)$ given by Eq. (33) is regular and positive in the whole interval $(-\infty, +\infty)$, moreover, $\sigma(\pm\infty) = 1$. Thus, $\sigma(r)$ given by Eq. (33) satisfies the necessary conditions (i-iii) and the corresponding field configuration can be considered as a candidate in searching for traversable wormholes. In the next section we will complete the solution for $\sigma(r)$ by the solution for $N(r)$ and discuss the properties of the non-minimal Wu-Yang wormhole solution.

IV. NON-MINIMAL WU-YANG WORMHOLE

In this section we consider in more details the special case corresponding to the following choice of the non-minimal coupling parameters q_1, q_2, q_3 :

$$q_1 = a^4/\kappa, \quad 12q_1 + 4q_2 + q_3 = 0. \quad (34)$$

Then, the equation (27) can be easily integrated in the quadratures to give

$$N(r) = \frac{(r^2 + a^2)^{3/2}}{r^3 \sqrt{r^2 + 2a^2}} \left\{ C + \int_0^r \frac{dx}{(x^2 + a^2)^{3/2} \sqrt{x^2 + 2a^2}} \left[x^4 + 2x^2 \left(a^2 - \frac{\kappa}{4} \right) - \left(10a^4 + \frac{\kappa a^2}{2} + 3\kappa q_2 \right) \right] \right\}, \quad (35)$$

where C is a constant of integration. Note that for arbitrary values of a, q_2 , and C the function $N(r)$ given by Eq. (35) satisfies the boundary condition $N(\pm\infty) = 1$. Near $r = 0$ the solution $N(r)$ is, generally speaking, divergent. Such behavior of $N(r)$ is unsuitable for description of traversable wormholes. However, there are special values of parameters q_2 and C , namely:

$$C = 0, \quad q_2 = -\frac{10a^4}{3\kappa} - \frac{a^2}{6}, \quad (36)$$

for which the solution (35) transforms into

$$N(r) = \frac{(r^2 + a^2)^{3/2}}{r^3 \sqrt{r^2 + 2a^2}} J(r), \quad (37)$$

where

$$J(r) = \int_0^r \frac{x^2 dx}{(x^2 + a^2)^{3/2} \sqrt{x^2 + 2a^2}} \left(x^2 + 2a^2 - \frac{\kappa}{2} \right) \quad (38)$$

is a function of r and two guiding parameters, a and κ . Note that near $r = 0$ the function $N(r)$, given by Eq. (37), behaves as

$$N(r) \simeq (3a^2)^{-1} (a^2 - \kappa/4) + O(r^2). \quad (39)$$

It is seen that $N(r)$ can be positive, negative, or zero at $r = 0$ depending on the relation between two parameters: a (the wormhole throat radius) and κ (the charge parameter). It will be convenient further to use a dimensionless parameter $\alpha = a\kappa^{-1/2}$. The behavior of $N(r)$ depending on α is illustrated in the Fig.1.

Taking into account the relations $g_{tt} = \sigma^2 N$ and $-g_{rr} = 1/N$ and using the solutions (33) and (37) for $\sigma(r)$ and $N(r)$ we finally obtain the following metric, which presents the new exact solution of the non-minimally extended Einstein-Yang-Mills equations:

$$ds^2 = \frac{(r^2 + a^2)^{5/2}}{r^3 (r^2 + 2a^2)^{3/2}} J(r) dt^2 - \frac{r^3 (r^2 + 2a^2)^{1/2}}{(r^2 + a^2)^{3/2}} \frac{dr^2}{J(r)} - (r^2 + a^2) (d\theta^2 + \sin^2 \theta d\varphi^2). \quad (40)$$

This metric describes a regular (i.e., without singularities) spacetime containing two asymptotically flat regions $r = \pm\infty$ connected by a throat located at $r = 0$. Thus, the metric (40) describes a wormhole, which we will hereafter call as a *non-minimal Wu-Yang wormhole*.

The spacetime structure of the Wu-Yang wormhole essentially depends on the value of the dimensionless parameter $\alpha = a\kappa^{-1/2}$. We note that for $\alpha > 1/2$ the function $N(r)$ is positive defined (see Fig. 1), and so the metric components $g_{tt} = \sigma^2 N$ and $-g_{rr} = 1/N$ are finite and positive in the whole region $(-\infty, +\infty)$. This means that the spacetime has no event horizons, thus in this case the Wu-Yang wormhole is *traversable*.

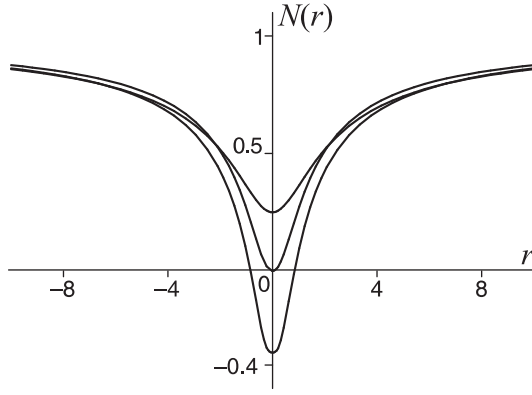


FIG. 1: Graphs of the function $N(r)$ given for $\alpha \equiv a\kappa^{-1/2} > 1/2$, $\alpha = 1/2$, and $\alpha < 1/2$ from up to down, respectively.

In case $\alpha < 1/2$ the function $N(r)$ changes the sign. It is positive for $|r| > r_h$, negative for $|r| < r_h$, and zero at $|r| = r_h$, i.e., $N(\pm r_h) = 0$ (r_h is some parameter, which can be easily found numerically for every $\alpha < 1/2$). In the vicinity of $|r| = r_h$ one has $g_{tt} \sim (r - r_h)$ and $g_{rr} \sim (r - r_h)^{-1}$. This means that the points $|r| = r_h$ are nothing but two event horizons of Schwarzschild-like type in the wormhole spacetime, and r_h is the radius of horizons. In the accepted nomenclature, the regions $|r| > r_h$ with $N(r) > 0$ and $|r| < r_h$ with $N(r) < 0$ are R- and T-regions, respectively. Thus, in the case $\alpha < 1/2$ the throat of Wu-Yang wormhole turns out to be hidden in the T-region behind the horizons. Such a wormhole is non-traversable from the point of view of a distant observer. By analogy with black holes one may call such objects as *black wormholes*.

Note that for $\alpha = 1/2$ two event horizons $|r| = r_h$ merge with each other and form an event horizon located at the wormhole's throat $r = 0$. Now, in the vicinity of $r = 0$ one has $g_{tt} \sim r^2$ and $g_{rr} \sim r^{-2}$, and this means that $r = 0$ is an extremal horizon. In this case the T-region is absent, and the event horizon divides two R-regions.

Now let us discuss a formula for an asymptotic mass of the Wu-Yang wormhole measured by a distant observer. A mass of a static spherically symmetric configuration is defined as $M = \frac{1}{2} \lim_{r \rightarrow \pm\infty} \{|r|(1 - g_{tt}(r))\}$. Using the metric (40) we can obtain after some algebra the following expression for the mass of the non-minimal Wu-Yang wormhole:

$$\frac{M}{\kappa^{1/2}} \equiv \widetilde{M}(\alpha) = \frac{\pi\sqrt{2\pi}}{\Gamma^2(\frac{1}{4})} \left(\alpha - \frac{1}{4\alpha} \right) + \frac{\Gamma^2(\frac{1}{4})}{16\sqrt{2\pi}} \frac{1}{\alpha}, \quad (41)$$

where $\Gamma(z)$ is gamma function, and $\alpha = a\kappa^{-1/2}$ and $\widetilde{M} = M\kappa^{-1/2}$ are dimensionless quantities. The graph of $\widetilde{M}(\alpha)$ is given in Fig. 2. It is worth to note that $\widetilde{M}(\alpha)$ is positive defined, $\widetilde{M}(\alpha) > 0$. Moreover, the function $\widetilde{M}(\alpha)$ has a minimum $\widetilde{M}_{\min} \approx 0.653$ at $\alpha = \alpha_{\min} \approx 0.545$.

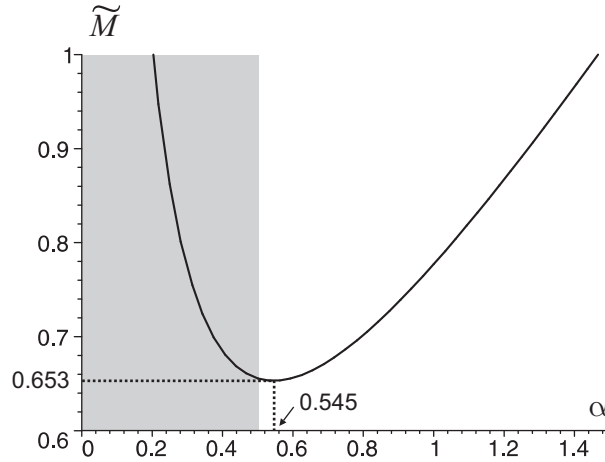


FIG. 2: Wormhole mass $\widetilde{M}(\alpha)$. The shaded region corresponds to $\alpha < 1/2$.

V. CONCLUSIONS

In this paper we have considered the non-minimally extended Einstein-Yang-Mills model given by the action (1). The model contains three phenomenological parameters q_1 , q_2 and q_3 , which determine the non-minimal coupling of the Yang-Mills and gravitational fields. In the framework of this model we have studied static spherically symmetric configurations with the Yang-Mills field possessing the SU(2) symmetry. Basing on the Wu-Yang ansatz for the gauge field we have obtained a three-parameter family of the explicit exact solutions to the non-linear Einstein-Yang-Mills equations. Only one solution from this family is regular and belongs to the class of wormhole spacetimes. We have denoted this solution as a *non-minimal Wu-Yang wormhole* (see Eq. (40)). Let us emphasize some of its properties.

1. The non-minimal Wu-Yang wormhole corresponds to the specific choice of coupling parameters q_1 , q_2 , q_3 , namely,

$$q_1 = \frac{a^4}{\kappa}, \quad q_2 = -\frac{10a^4}{3\kappa} - \frac{a^2}{6}, \quad q_3 = \frac{4a^4}{3\kappa} + \frac{2a^2}{3}. \quad (42)$$

Thus, the Wu-Yang wormhole geometry turns out to be completely determined by two model parameters: the wormhole throat radius a , and the charge parameter $\kappa = 8\pi\nu^2/\mathcal{G}^2$, or, equivalently, by a and the dimensionless parameter $\alpha \equiv a\kappa^{-1/2}$. Note that in the minimal limit, when $q_1 = q_2 = q_3 = 0$, the relations (42) yield $a = 0$, i.e., this wormhole does not exist. In other words, the obtained exact solution is *essentially* non-minimal.

2. The parameter α can be treated as *guiding* one. Indeed, in case $\alpha > 1/2$ the spacetime of Wu-Yang wormhole has no event horizons, and so it is traversable in principle. The condition $\alpha > 1/2$ equivalently reads $a > \frac{1}{2}\kappa^{1/2}$, that is the throat's radius a of traversable Wu-Yang wormholes is necessary greater than $\frac{1}{2}\kappa^{1/2}$. In case $\alpha < 1/2$ ($a < \frac{1}{2}\kappa^{1/2}$) the wormhole spacetime (40) possesses two Schwarzschild-type event horizons at $|r| = r_h$, where r_h is an event horizon radius given by the equation $\sigma^2 N(r_h) = 0$. The presence of event horizons means the Wu-Yang wormhole is non-traversable from the point of view of a distant observer. It is worth to note that in this case the wormhole throat located at $r = 0$ turns out to be hidden behind the horizons. For this reason one can call such objects as *black wormholes*. For the particular value $\alpha = 1/2$ ($a = \frac{1}{2}\kappa^{1/2}$) two event horizons merge with each other and form a single event horizon at the throat $r = 0$. Now in the vicinity of $r = 0$ the metric functions behave as $g_{tt} \sim r^2$ and $g_{rr} \sim r^{-2}$, and so the metric (40) behaves near the horizon as the extreme Reissner-Nordström metric.
3. For a distant observer the Wu-Yang wormhole manifests itself through its asymptotical mass M . It is determined by the charge parameter κ and expressed through the wormhole throat radius a (see Eq. (41) and Fig. 2). Is it possible for the observer to reconstruct the invisible throat radius using the estimated mass? In principle, yes, but the procedure is ambiguous, since two values of a correspond to one appropriate value of the mass.
4. The important feature is that there exists the *lower limit* for the mass of the non-minimal Wu-Yang wormhole. In other words, the wormhole mass cannot be less than some minimal value $M_{\min} \approx 0.653 \kappa^{1/2}$, i.e., $M \geq M_{\min}$. To make estimations we assume that the monopole magnetic charge ν is equal to one, $\nu = 1$, and the square of the constant of gauge interaction is given by $\mathcal{G}^2 = 4\pi\alpha_{\text{em}}$, where $\alpha_{\text{em}} = e^2/\hbar c \approx 1/137$ is the fine structure constant. Then, in the dimensional units we have $M_{\min} \approx 10.8 M_{\text{pl}}$, $a_{\min} \approx 9 L_{\text{pl}}$, where M_{pl} and L_{pl} are the Planck mass and the Planck length, respectively.

Recently Kirill Bronnikov attracted our attention to the papers [24, 25], where the authors discuss solutions they refer to as regular black holes. He also emphasized that black wormholes obtained in our paper represent the kind of regular black holes.

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